Further Pure 3

Topic 2: Differential Equations: Summary Information

A differential equation is an equation connecting a function and its derivatives.

A first-order differential equation involves only first derivatives. Examples of first-order differential equations are

$$\frac{dy}{dx} + 2xy = 3x^2$$

$$(3x-2)\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} + 3y = 2x - 1$$

The first and third of these are examples of linear differential equations as the y and $\frac{dy}{dx}$ terms appear only in the first degree (they are not squared or cubed).

A second-order differential equation involves second derivatives. Examples of second-order differential equations are

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{4x}$$

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

These are both linear differential equations.

METHODS FOR SOLVING FIRST-ORDER DIFFERENTIAL EQUATIONS

Method 1: Finding the complementary function and a particular solution.
This method is suitable for solving only linear first-order differential equations.

Method 2: Separating the variables (as in C4)
This method only works for differential equations which can be rearranged to the form

$$f(x)\frac{dy}{dx} = g(y).$$

Method 3: Integrating factors

This method works if the differential equation can be rearranged to the form $\frac{dy}{dx} + P(x)y = Q(x)$.

LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

Suppose that the differential equation has the form $a\frac{dy}{dx} + by = f(x)$.

Step 1: Find the **complementary function**, that is the solution to the equation $a\frac{dy}{dx} + by = 0$.

The complementary function is $y = Ae^{mx}$ where m is the solution to the **auxiliary equation**, $am + b = 0$

Step 2: Find the **particular solution**, that is a function that satisfies the original differential equation.

Step 3: The general solution to the original differential equation is
y = complementary function + particular solution

EXAMPLE:

Find the general solution to the differential equation $2\frac{dy}{dx} - 3y = e^{2x}$.

Solution:

The auxiliary equation is $2m - 3 = 0$.

$$\Rightarrow m = 1.5$$

Therefore the complimentary function is $y = Ae^{1.5x}$.

Try $y = ae^{2x}$ as a particular solution.

$\Rightarrow \frac{dy}{dx} = 2ae^{2x}$

Substituting these into the original differential equation gives:

$$2 \times 2ae^{2x} - 3ae^{2x} = e^{2x}$$

$$i.e.\ \ ae^{2x} = e^{2x}$$

$$i.e.\ a = 1$$

Therefore the particular solution is $y = e^{2x}$.

So, the general solution to the original differential equation is $y = Ae^{1.5x} + e^{2x}$.

DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

Suppose that the differential equation has the form $g(y)\frac{dy}{dx} = f(x)$ ..

The variables can then be separated out: $\int g(y)dy = \int f(x)dx$

EXAMPLE:

Find the solution to the differential equation $x^2 \frac{dy}{dx} = (x+1)(y+1)$ given that $y = 2e$ when $x = 1$.

SOLUTION:

We first separate out the variables: $x^2 dy = (x+1)(y+1)dx$

$$\frac{1}{y+1}dy = \frac{x+1}{x^2}dx$$

Put in integration signs: $\int \frac{1}{y+1}dy = \int \frac{x+1}{x^2}dx$

Evaluating the right hand side: $\int \frac{x+1}{x^2}dx = \int \left(\frac{x}{x^2} + \frac{1}{x^2}\right)dx = \int \left(\frac{1}{x} + x^{-2}\right)dx$

$$= \ln x - x^{-1} + c$$

Evaluating the right hand side: $\int \frac{1}{y+1}dy = \ln(y+1)$

Put together: $\ln(y+1) = \ln x - x^{-1} + c$

Take exponentials of both sides: $y+1 = e^{\ln x - x^{-1} + c}$

$$y+1 = e^{\ln x} e^{x^{-1}} e^{c}$$

$$y = Axe^{x^{-1}}$$

Substitute in $y = 2e$ and $x = 1$:

$$2e = Ae^{1}$$

Therefore $A = 2$

So the solution is $y = 2xe^{x^{-1}}$

SOLVING DIFFERENTIAL EQUATIONS WITH INTEGRATING FACTORS

This method works if the differential equation can be rearranged to the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 1: Rearrange the differential equation to the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 2: Find the integrating factor: $I(x) = e^{\int P(x)dx}$.

Step 3: Multiply through by the integrating factor: $I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$. This can be written as $\frac{d}{dx}\left(I(x)y\right) = I(x)Q(x)$.

Step 4: The solution therefore is $I(x)y = \int I(x)Q(x)dx$.

EXAMPLE:

Given that -1 < x < 1, find the general solution of the differential equation $(1-x^2)\frac{dy}{dx} - xy + 1 = 0$.

Find the general solution for which $y = \frac{\pi}{2}$ when x = 0.

SOLUTION:

First we get the differential equation into the correct form by dividing by $1-x^2$:

$$\frac{dy}{dx} - \frac{x}{(1-x^2)}y = \frac{-1}{1-x^2} \qquad (*)$$

We therefore see that $P(x) = \frac{-x}{1-x^2}$

Next we find $\int P(x)dx = \int \frac{-x}{1-x^2}dx$ $\qquad u = 1-x^2$

$= \int \frac{1/2}{u}du = \frac{1}{2}\ln u$ $\qquad \frac{du}{dx} = -2x$

$= \ln(1-x^2)^{1/2}$ $\qquad \frac{1}{2}du = -xdx$

The integrating factor is $I(x) = e^{\int P(x)dx} = e^{\ln(1-x^2)^{1/2}} = \sqrt{1-x^2}$

Multiply equation (*) by $I(x)$: $\qquad \sqrt{1-x^2}\frac{dy}{dx} - \frac{x\sqrt{1-x^2}}{(1-x^2)}y = \frac{-\sqrt{1-x^2}}{1-x^2}$

This simplifies to $\sqrt{1-x^2}\frac{dy}{dx} - \frac{x}{\sqrt{1-x^2}}y = \frac{-1}{\sqrt{1-x^2}}$.

This is equivalent to $\frac{d}{dx}\left(y\sqrt{1-x^2}\right) = \frac{-1}{\sqrt{1-x^2}}$ $\qquad$ i.e. $y\sqrt{1-x^2} = \int \frac{-1}{\sqrt{1-x^2}}dx$

Using tables we get: $y\sqrt{1-x^2} = -\sin^{-1}x + c$

Substitute $y = \frac{\pi}{2}$ when x = 0: $\frac{\pi}{2} = 0 + c$ i.e $c = \pi/2$.

Therefore the overall solution is $y\sqrt{1-x^2} = -\sin^{-1}x + \frac{\pi}{2}$.

USING SUBSTITUTIONS TO TRANSFORM A DIFFERENTIAL EQUATION

Sometimes it is possible to solve more complex differential equations by changing the variables using a substitution.

EXAMPLE:

Show that the differential equation $\frac{dy}{dx} = \frac{x^2 - y^2}{y}$ can be transformed into a linear differential equation by means of the substitution $y = \sqrt{z}$. Hence find the general solution of the equation.

SOLUTION:

$$y = \sqrt{z}$$
$$\frac{dy}{dz} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{2}z^{-1/2}\frac{dz}{dx}$$

The equation $\frac{dy}{dx} = \frac{x^2 - y^2}{y}$ becomes $\frac{1}{2}z^{-1/2}\frac{dz}{dx} = \frac{x^2 - z}{\sqrt{z}}$.

Multiply through by $\sqrt{z}$: $\frac{1}{2}\frac{dz}{dx} = x^2 - z$.

So we get: $\frac{dz}{dx} + 2z = 2x^2$ (a linear differential equation).

The auxiliary equation is $m + 2 = 0$ i.e. $m = -2$.
So the complementary function is $z = Ae^{-2x}$.

Try $z = ax^2 + bx + c$ as a particular solution. Then $\frac{dz}{dx} = 2ax + b$.
Substitute into the differential equation to get:

$$(2ax + b) + 2(ax^2 + bx + c) = 2x^2$$

Comparing coefficients we see that:

$a = 1$ (coefficients of x^2)

$2a + 2b = 0$ (coefficients of x)
i.e. $b = -1$

$b + 2c = 0$ (coefficients of units)
$c = ½$

Therefore a particular solution is $z = x^2 - x + 0.5$

Hence the solution to the differential equation is $z = Ae^{-2x} + x^2 - x + 0.5$

So, $y = \sqrt{z} = \sqrt{Ae^{-2x} + x^2 - x + 0.5}$.

SOLVING SECOND-ORDER DIFFERENTIAL EQUATIONS

In FP3, we need to solve second-order differential equations with the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = f(x).$$

STEP 1: Find the complementary function.
This is the solution to the reduced equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = 0.$$

We find the complementary function by solving the auxiliary equation, which is the quadratic:

$$am^2+bm+c=0.$$

There are three cases to consider.

Case 1: The auxiliary equation has two distinct, real roots. If these roots are m_1 and m_2 then the complementary function is $y = Ae^{m_1x}+Be^{m_2x}$.

Case 2: The auxiliary equation has a repeated real root. If this repeated root is m_1, then the complementary function is $y=(Ax+B)e^{m_1x}$.

Case 3: The auxiliary equation has complex roots, $\alpha \pm i\beta$. Then the complementary function is $y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$.

STEP 2: Find the particular solution.
The particular solution is any function $y=P(x)$ which solves the original differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = f(x)$.
The choice of particular solution depends on the nature of the function on the right-hand side of this differential equation.

$f(x) =$	Choice for particular solution
constant	constant, c
linear function	linear function, $ax + b$
quadratic function	quadratic function, ax^2+bc+c
exponential function $f(x)=ke^{px}$	exponential function, ae^{px}
trigonometric function $f(x)=k\sin px$ or $f(x)=k\cos px$	Trigonometric function $a\cos px+b\sin px$

STEP 3: Find the general solution.
The general solution of the original differential equation is formed by ADDING the complementary function and the particular solution together.

STEP 4: Use the boundary conditions
If you are given boundary conditions (i.e. values for y or $\frac{dy}{dx}$ corresponding to given x values) you can substitute these into the general solution in order to find the values of any unknown constants.

Further note:

If the natural choice for the particular solution already forms part of the complementary function, you need to adjust it by multiplying it by *x*. Mathematically, this means that if the usual form for the particular solution would be $y = F(x)$, but this forms part of the complementary function, you would instead use $y = xF(x)$. This technique will be illustrated in example 3 below.

EXAMPLE 1:

Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 50\sin 2x.$$

Given that $y = 0$ when $x = 0$, and that y remains finite as $x \to \infty$, find y in terms of x.

SOLUTION:

Reduced equation: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

Auxiliary equation: $m^2 - 3m - 4 = 0$

$(m+1)(m-4) = 0$

$m = -1$ or $m = 4$

Complementary function is $y = Ae^{-x} + Be^{4x}$

Try $y = a\cos 2x + b\sin 2x$ as the particular solution.

Then $\frac{dy}{dx} = -2a\sin 2x + 2b\cos 2x$ and $\frac{d^2y}{dx^2} = -4a\cos 2x - 4b\sin 2x$

Substituting these into the original differential equations gives:

$$(-4a\cos 2x - 4b\sin 2x) - 3(-2a\sin 2x + 2b\cos 2x) - 4(a\cos 2x + b\sin 2x) = 50\sin 2x$$

Comparing coefficients:

$\cos 2x$:	$-4a - 6b - 4a = 0$	i.e.	$8a = -6b$
$\sin 2x$:	$-4b + 6a - 4b = 50$	i.e.	$-8b + 6a = 50$
		i.e.	$-8\left(\frac{-8}{6}a\right) + 6a = 50$
		i.e.	$a = 3$
		So	$b = -4$

So the particular solution is $y = 3\cos 2x - 4\sin 2x$.

So the general solution is $y = Ae^{-x} + Be^{4x} + 3\cos 2x - 4\sin 2x$.

<u>Boundary conditions:</u>

Substituting $x = 0$, $y = 0$ gives: $0 = A + B + 3$ i.e. $A + B = -3$

Next consider what happens as $x \to \infty$

$e^{-x} \to 0$

$3\cos 2x$ remains finite

$4\sin 2x$ remains finite

but $e^{4x} \to \infty$

As we are told that y remains finite as $x \to \infty$, we must have that $B = 0$. Therefore $A = -3$.

So the overall solution is $y = -3e^{-x} + 3\cos 2x - 4\sin 2x$.

EXAMPLE 2:

Find the general solution for the differential equation $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 20x = 100t$.

Find the solution corresponding to the initial conditions $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

SOLUTION:

Reduced equation: $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 20x = 0$

Auxiliary equation: $m^2 - 4m + 20 = 0$

$$m = \frac{4 \pm \sqrt{16 - 4 \times 20}}{2} = \frac{4 \pm 8i}{2} = 2 \pm 4i$$

Therefore the complementary function is: $x = e^{2t}(A\cos 4t + B\sin 4t)$

As the right hand side of the original differential equation is a linear function, we choose to use a linear function as the particular solution.

Let $x = at + b$

Then $\frac{dx}{dt} = a$

and $\frac{d^2x}{dt^2} = 0$

Substituting these into the differential equation gives:

$$0 - 4a + 20(at + b) = 100t$$

Comparing coefficients:

Coefficients of t: $20a = 100$ i.e. $a = 5$

Coefficients of units: $-4a + 20b = 0$

$-20 + 20b = 0$

$b = 1$.

Therefore $x = 5t + 1$ is a particular solution.

So the general solution is $x = e^{2t}(A\cos 4t + B\sin 4t) + 5t + 1$

Boundary conditions:

Substitute in $x = 0$, $t = 0$: $0 = 1(A) + 1$ i.e. $A = -1$

Now we need to differentiate to find $\frac{dx}{dt}$:

$$\frac{dx}{dt} = 2e^{2t}(A\cos 4t + B\sin 4t) + e^{2t}(-4A\sin 4t + 4B\cos 4t) + 5$$

Substitute in $\frac{dx}{dt} = 0$, $t = 0$: $2(A) + 1(4B) + 5 = 0$

$-2 + 4B + 5 = 0$

$B = ¾$

So the overall solution is $x = e^{2t}(-\cos 4t + \frac{3}{4}\sin 4t) + 5t + 1$

EXAMPLE 3:

Consider the differential equation $\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 24y = 4e^{2x}$.

Find y in terms of x, given that $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$.

SOLUTION:

Reduced equation: $\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 24y = 0$

Auxiliary equation: $m^2 - 14m + 24 = 0$

$(m - 12)(m - 2) = 0$

$m = 12$ or $m = 2$

So the complementary function is $y = Ae^{12x} + Be^{2x}$

The right hand side of the differential equation is $4e^{2x}$. We would therefore usually try $y = ae^{2x}$ as a particular solution, but this already forms part of the complementary function. Instead, we try $y = axe^{2x}$.

$$\frac{dy}{dx} = ae^{2x} + 2axe^{2x} \qquad \text{(using the product rule)}$$

$$\frac{d^2y}{dx^2} = 2ae^{2x} + 2ae^{2x} + 4axe^{2x}$$

Substituting into the original differential equation $\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 24y = 4e^{2x}$ gives:

$$\left(2ae^{2x} + 2ae^{2x} + 4axe^{2x}\right) - 14\left(ae^{2x} + 2axe^{2x}\right) + 24axe^{2x} = 4e^{2x}$$

If we simplify the LHS, we get:

$$-10ae^{2x} = 4e^{2x}$$

So $a = -0.4$

Therefore the particular solution is $y = -0.4xe^{2x}$

So the general solution is $y = Ae^{12x} + Be^{2x} - 0.4xe^{2x}$.

Boundary conditions:

Substitute in $y = 0$, $x = 0$: $0 = A + B$ i.e. $A = -B$

To use the second boundary condition, we need to differentiate:

$$\frac{dy}{dx} = 12Ae^{12x} + 2Be^{2x} - 0.4e^{2x} - 0.8xe^{2x}$$

Substitute in $\frac{dy}{dx} = 0, x = 0$: $0 = 12A + 2B - 0.4$

$0.4 = 12(-B) + 2B$

$B = -0.04$

$A = 0.04$

So $y = 0.04e^{12x} - 0.04e^{2x} - 0.4xe^{2x}$.